Complete Lift and Metrics on the Complex Cotangent Bundle

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1. **Introduction:** Let M_{2n} be 2n-dimensional differential manifold of the class C^{∞} and T^2M_{2n} be its cotangent bundle. K.P. Mok [2] and several authors have studied metrics and connections on the cotangent bundle and tangent bundle, but no natural conjecture has been presented for the study of metrics and connection of the complex structure on the cotangent bundle. Almost complex product manifold, the Riemann extensions and the complete lift and the metric $T^2(g,\nabla)$ have been studied in section 2, section 3 and section 4 respectively.
2. **Complex cotangent bundle as an almost complex product manifold**:

Let M_{2n} be an 2n-dimensional real manifold of class C^{∞} and $T^2{}_pM_{2n}$ the cotangent space at the point $p\in M_{2n}$, then the set

$$T^2\left(M_{2n}\right)=\bigcup_{p\in M_{2n}} T^2{}_p\left(M_{2n}\right)$$

is called complex cotangent bundle over M_{2n}.

For any point $\sigma\in M_{2n}$, the mapping $\sigma\rightarrow p$ determines the bundle projection defined by $\pi:T^2M_{2n}\rightarrow M_{2n}$. Let V be the field of 2n complex planes on T^2M_{2n}. It is an integrable distribution on T^2M_{2n} which we call the vertical distributions uniquely on T^2M_{2n} an 2n-dimentional distribution complementary to T^2M_{2n}, which we call the horizontal distribution and is denoted by H, the pair (H,V) defines an almost complex product manifold. Let $\left\{\pi^{-1}U\left((z^{\alpha},v_{\alpha}),(z^{\bar{\alpha}},v_{\bar{\alpha}})\right)\right\}$ be an induced co-ordinates to $\pi^{-1}(U)$ is spanned by 2n independent vector fields.

(2.1) $$D_s=\frac{\partial}{\partial s^s}+\Gamma_{sr}\frac{\partial}{\partial p_r}$$

$$D_{\bar{s}}=\frac{\partial}{\partial s^{\bar{s}}}+\Gamma_{\bar{s}\bar{r}}\frac{\partial}{\partial p_{\bar{r}}}$$

(2.2) Where $\Gamma_{sr}=p_{\alpha}\Gamma^{\alpha}_{sr}$, $\Gamma_{\bar{s}\bar{r}}=p_{\bar{\alpha}}\Gamma^{\bar{\alpha}}_{\bar{s}\bar{r}}$

The vertical distribution V restricted to $\pi^{-1}(U)$ is spanned by 2n independent vector fields.

(2.3) $$D_s=\frac{\partial}{\partial s^s},D_{\bar{s}}=\frac{\partial}{\partial s^{\bar{s}}}$$

It follows that $\{D_s\}$ and $\{D_{\bar{s}}\}$ constitutes a frame on $\pi^{-1}(U)$. As the frame is adopted to he almost complex structure (H,V), we call them adopted frames on $\pi^{-1}(U)$ and is components are called frame components on $\pi^{-1}(U)$ dual to the adopted frame are given by

(2.4) $D^{s} = dx^{s}, D^{\bar{s}} = dx^{\bar{s}}$

(2.5)

$$D^{s} = -\Gamma_{sr} dx^{r} + dp_{s}$$

$$D^{\bar{s}} = -\Gamma_{\bar{s}\bar{r}} dx^{\bar{r}} + dp_{\bar{s}}$$

The component matrix of the adopted frame and its co-frame are

(2.6)

$$\left[L_{\beta}^{A}\right] = \begin{bmatrix} \delta_{rs} & 0 & 0 & 0 \\ \Gamma_{sr} & \delta_{rs} & 0 & 0 \\ 0 & 0 & \delta_{\bar{r}\bar{s}} & 0 \\ 0 & 0 & \Gamma_{\bar{r}\bar{s}} & \delta_{\bar{s}\bar{r}} \end{bmatrix}$$

and

(2.7)

$$L^{-1} = \left[L_{s}^{r}\right] = \begin{bmatrix} \delta_{sr} & 0 & 0 & 0 \\ -\Gamma_{sr} & \delta_{rs} & 0 & 0 \\ 0 & 0 & \delta_{\bar{s}\bar{r}} & 0 \\ 0 & 0 & -\Gamma_{\bar{s}\bar{r}} & \delta_{\bar{s}\bar{r}} \end{bmatrix}$$

respectively.

The non holomorphic of adopted frame are given by

(2.8)

$$\Omega_{\alpha\beta}^{\gamma} = \left[D_{\alpha}(L_{\beta}^{A}) - D_{\beta}(L_{\alpha}^{A})L_{\beta}^{A}\right]$$

$$\Omega_{\bar{\alpha}\bar{\beta}}^{\bar{\gamma}} = \left[D_{\bar{\alpha}}(L_{\bar{\beta}}^{\bar{A}}) - D_{\bar{\beta}}(L_{\bar{\alpha}}^{\bar{A}})L_{\bar{\beta}}^{\bar{A}}\right]$$

Using (2.1), (2.3),(2.6),(2,7) and (2.8), we get possible non zero components of $\Omega_{\alpha\beta}^{\gamma}$ as

(2.9) $\Omega_{\alpha\beta}^{h} = p_{m} R_{srh}^{m}$

$, \Omega_{\bar{\alpha}\bar{\beta}}^{\bar{h}} = p_{\bar{m}} R_{\bar{s}\bar{r}\bar{h}}^{\bar{m}}$

The projective tensor of $T^{2}M_{2n}$ onto H and V again denoted by H and V. They are tensor of type (1, 1) on $T^{2}M_{2n}$ whose frame component matrix are given by

$$H = \begin{bmatrix} \delta_{sr} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \delta_{sr} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(2.10)

$$V = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \delta_{sr} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_{\bar{s}\bar{r}} \end{bmatrix}$$

$$(2.11) H^{2} = H, V^{2} = V, HV = VH = 0, H + V = scalarMatrix$$

and torsion tensor $S = S_{H,V}$ associated with H and V reduces by virtue of (2.11) and intergrability of V is

(2.12) $S(\tilde{P}, \tilde{Q}) = -2V(\tilde{HP}, \tilde{HQ})$

Where $\tilde{P}$ and $\tilde{Q}$ are arbitrary vector fields on $T^{2}M_{2n}$ and possible non zero frame components of S are

(2.13) $$S^h_{sr} = -2 p_m R^m_{srh}$$

$$S^{\bar{h}}_{\bar{s}\bar{r}} = -2 p_{\bar{m}} R^{\bar{m}}_{\bar{s}\bar{r}\bar{h}}$$

Let $\tilde{\nabla}$ be an arbitrary linear connection on $T^2 M_{2n}$ whose components in $\pi^{-1}(U)$ are $\tilde{\Gamma}^A_{CB}$ and $\tilde{\Gamma}^{\bar{A}}_{\bar{C}\bar{B}}$. The frame components of $\tilde{\nabla}$ in $\pi^{-1}(U)$ are defined by

(2.14)

$$\tilde{\Gamma}^{\gamma}_{\beta\alpha} = \left[D_s \left(L^A_B \right) + \tilde{\Gamma}^{\gamma}_{\beta\alpha} L^C_{\alpha} L^B_{\beta} \right] L^{\gamma}_A$$

$$\tilde{\Gamma}^{\bar{\gamma}}_{\beta\alpha} = \left[D_s \left(L^{\bar{A}}_{\bar{\beta}} \right) + \tilde{\Gamma} L^{\bar{C}}_{\bar{C}\bar{B}} L^{\bar{B}}_{\bar{\beta}} \right] L^{\bar{\gamma}}_{\bar{A}}$$

If $\tilde{X}$ is a vector field on $T^2 M_{2n}$ whose frame components are $\tilde{X}^V$ then

(2.15) $$\tilde{\nabla} \tilde{X}^V = D_{\alpha}(\tilde{X}^V) + \tilde{\Gamma}^{\gamma}_{\beta h} \tilde{X}^h$$

$$\overline{\nabla \tilde{X}} = D_{\bar{\alpha}}(\overline{\tilde{X}}) + \tilde{\Gamma}^{\bar{\gamma}}_{\bar{\beta}\bar{h}} \tilde{X}^{\bar{h}}$$

The frame component of the torsion tensor and curvature tensor of $\tilde{\nabla}$ is given by

(2.16) $$\tilde{T}^{\gamma}_{\beta\alpha} = \tilde{\Gamma}^{\gamma}_{\beta\alpha} - \tilde{\Gamma}^{\gamma}_{\alpha\beta} \Omega_{\beta\gamma}$$

$$\tilde{T}^{\bar{\gamma}}_{\bar{\beta}\bar{\alpha}} = \tilde{\Gamma}^{\gamma}_{\beta\alpha} - \tilde{\Gamma}^{\bar{\gamma}}_{\bar{\alpha}\bar{\beta}} \Omega_{\bar{\beta}}$$

And

$$\tilde{R}^{\gamma}_{p\alpha\beta} = D_p\left(\tilde{R}^{\gamma}_{\beta\alpha}\right) - D_{\alpha}\left(\tilde{R}^{\gamma}_{p\beta}\right) + \tilde{\Gamma}^{\gamma}_{p\in} \tilde{\Gamma}^{\gamma}_{\beta\alpha} - \tilde{\Gamma}^{\gamma}_{\alpha\beta} \tilde{\Gamma}^{\gamma}_{p\beta} - \Omega^{\gamma}_{p\alpha} \tilde{\Gamma}^{\gamma}_{p\beta} \gamma$$

$$\tilde{R}^{\bar{\gamma}}_{p\alpha\beta} = D_{\bar{p}}\left(\tilde{R}^{\bar{\gamma}}_{\bar{\beta}\bar{\alpha}}\right) - D_{\bar{\alpha}}\left(\tilde{R}^{\bar{\gamma}}_{\bar{p}\bar{\beta}}\right) + \tilde{\Gamma}^{\bar{\gamma}}_{\bar{p}} \tilde{\Gamma}^{\bar{\gamma}}_{\bar{p}\bar{\beta}} - \tilde{\Gamma}^{\bar{\gamma}}_{\bar{p}\bar{\beta}} \tilde{\Gamma}^{\bar{\gamma}}_{\bar{p}\bar{\beta}} - \Omega^{\bar{\gamma}}_{\bar{p}\bar{\beta}} \tilde{\Gamma}^{\bar{\gamma}}_{\bar{p}\bar{\beta}} \bar{r}$$

3. The Riemannian extension and complete lift of complex cotangent bundle

Let ∇ be free linear connection on M_{2n}.

If Γ^h_{sr} and $\Gamma^{\bar{h}}_{\bar{s}\bar{r}}$ are component of $\pi^{-1}U$ in U of v. The component matrix in v of the Riemannien extension is

(3.1)

$$\begin{bmatrix} -2\Gamma_{sr} & \delta_{sr} & 0 & 0 \\ \delta_{sr} & 0 & 0 & 0 \\ 0 & 0 & -2\Gamma_{\bar{s}\bar{r}} & \delta_{\bar{s}\bar{r}} \\ 0 & 0 & \delta_{\bar{s}\bar{r}} & 0 \end{bmatrix}$$

The corresponding frame component matrix is

(3.2)

$$\left[G_{\alpha\beta} \right] = \begin{bmatrix} 0 & \delta_{sr} & 0 & 0 \\ \delta_{sr} & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_{\bar{s}\bar{r}} \\ 0 & 0 & \delta_{\bar{s}\bar{r}} & 0 \end{bmatrix}$$

Let ∇^c be the Riemannian connection $T^2 M_{2n}$ associated with Riemann extension called the complete lift of ∇ to T^2M_{2n}. The frame components $\Gamma^{\gamma}_{\alpha\beta}$, $\Gamma^{\bar{\gamma}}_{\bar{\alpha}\bar{\beta}}$ of its associated Riemann connection are given by

(3.3)

$$\Gamma^{\gamma}_{\alpha\beta} = \frac{1}{2}G^{p\alpha}\left(D_{\alpha}G_{\alpha\beta} + D_{\beta}G_{p\alpha} - D_{p}G_{\beta\alpha}\right) + \frac{1}{2}\left(\Omega^{\gamma}_{\alpha\beta} + \Omega^{\gamma}_{\beta\alpha} + \Omega^{\gamma}_{p\beta}\right)$$

$$\bar{\Gamma}^{\gamma}_{\alpha\beta} = \frac{1}{2}G^{\bar{\gamma}\bar{\alpha}}\left(D_{\bar{\alpha}}G_{\bar{p}\bar{\beta}} + D_{\bar{\beta}}G_{\bar{\beta}\bar{\alpha}} - D_{\bar{p}}G_{\bar{\beta}\bar{\alpha}}\right) + \frac{1}{2}\left(\Omega^{\gamma}_{\alpha\beta} + \Omega^{\gamma}_{\beta\alpha} + \Omega^{\gamma}_{p\beta}\right)$$

Where $\left[G_{\alpha\beta}\right]$ is the inverse of $\left[G^{\alpha\beta}\right]$ and

(3.4)

$$\Omega^{\gamma}_{\alpha\beta} = G^{\gamma p}G_{\beta p}\Omega^{\gamma}_{p\varepsilon}, \Omega^{\bar{\kappa}}_{\bar{\alpha}\bar{\beta}} = G^{\bar{\gamma}\bar{p}}G_{\bar{\beta}\bar{\gamma}}\Omega^{\bar{\gamma}}_{\bar{p}\bar{\alpha}}$$

Using (3.3), we get possible non zero frame components of the complete lift ∇^c as

(3.5) $$\tilde{\Gamma}^{h}_{sr} = \Gamma^{h}_{sr}, \bar{\tilde{\Gamma}}^{h}_{sr} = \Gamma^{\bar{h}}_{\bar{s}\bar{r}}$$

$$\tilde{\Gamma}^{h}_{sr} = -\Gamma^{r}_{sh}, \bar{\tilde{\Gamma}}^{h}_{sr} = \Gamma^{\bar{r}}_{\bar{s}\bar{h}}$$ and

(3.6) $$\Gamma^{\bar{h}}_{\bar{s}\bar{r}} = p_{\alpha}R^{a}_{hrs}$$

$$\bar{\tilde{\Gamma}}^{\bar{h}}_{\bar{s}\bar{r}} = p_{\bar{\alpha}}R^{\bar{a}}_{\bar{h}\bar{r}\bar{s}}$$

By using (3.6) and (2.16), we get the possible non zero components of the curvature tensor $\tilde{R}$ of ∇^c as

(3.7) $$\tilde{R}^{h}_{ksr} = R^{h}_{ks}; \bar{\tilde{R}}^{h}_{ksr} = R^{\bar{h}}_{\bar{k}\bar{s}\bar{r}}$$

$$\tilde{R}^{h}_{ksr} = p_{\alpha}\left(\nabla_{k}R^{a}_{hrs} - \nabla_{s}R^{a}_{hrk}\right)$$

$$\bar{\tilde{R}}^{h}_{ksr} = p_{\bar{a}}\left(\nabla_{\bar{k}}R^{\bar{a}}_{\bar{h}\bar{r}\bar{s}} - \nabla_{s}R^{\bar{a}}_{\bar{h}\bar{r}\bar{k}}\right)$$

$$\tilde{R}^{h}_{ksr} = -R^{r}_{ksh}$$

$$\bar{\tilde{R}}^{h}_{ksr} = -R^{\bar{r}}_{\bar{k}\bar{s}\bar{h}}$$

$$\tilde{R}^{h}_{ksr} = -R^{s}_{hrk}$$

$$\bar{\tilde{R}}^{h}_{ksr} = -R^{\bar{s}}_{\bar{h}\bar{r}\bar{k}}$$

From (3.7) we have

Proposition 3.1: Let ∇ be a torsion free linear connection on $T^2M_{2n,}$, then (T^2M_{2n}, ∇^c) is locally flat if (M_{2n}, ∇) is locally flat.

4. The metric $T^2(g, \nabla)$:

Let g be a metric and ∇ be torsion free linear connection on $T^2M_{2n,}$, which we call $T^2(g,\nabla)$.The line element of $T^2(g,\nabla)$ on $\pi^{-1}(U)$ are

$$g_{rs}dx^{r}dx^{s} + g^{rs}\delta p_{s}\delta p$$

(4.1) $$g_{\bar{r}\bar{s}}dx^{\bar{r}}dx^{\bar{s}} + g^{\bar{r}\bar{s}}\delta p_{\bar{s}}\delta p_{\bar{r}}$$

Where $\delta p_{s} = dp_{s} - p_{a}\Gamma^{a}_{sr}dx^{r}$;

$$\bar{\delta} p_{s} = dp_{\bar{s}} - p_{\bar{a}}\Gamma^{\bar{a}}_{\bar{s}\bar{r}}dx^{\bar{r}}$$

Are the usual covariant differentiation (4.1) defines a global metric on $T^2M_{2n,}$ and its frame component matrix of $T^2(g,\nabla)$ is

(4.2)

$$\left[G_{\alpha\beta}\right] = \begin{bmatrix} g_{sr} & 0 & 0 & 0 \\ 0 & g^{sr} & 0 & 0 \\ 0 & 0 & g_{\bar{s}\bar{r}} & 0 \\ 0 & 0 & 0 & g^{\bar{s}\bar{r}} \end{bmatrix}$$

Let g be the metric tensor of $T^2(g,\nabla)$, by a simple calculation the possible non zero frame components of ∇ are given by

(4.3)
$$\nabla^c_k G_{sr} = \nabla_k g_{sr}$$
$$\nabla^{\bar{c}}_{\bar{k}} G_{\bar{s}\bar{r}} = \nabla_{\bar{k}} g_{\bar{s}\bar{r}}$$
$$\nabla^c_k G_{sr} = \nabla^c_k g_{rs}$$
$$\nabla^c_{\bar{k}} G_{\bar{s}\bar{r}} = \nabla_{\bar{k}} g_{\bar{s}\bar{r}}$$
$$\nabla^c_k G_{sr} = \nabla_k g^{sr}$$
$$\nabla^{\bar{c}}_{\bar{k}} G_{\bar{s}\bar{r}} = \nabla_{\bar{k}} g^{\bar{s}\bar{r}}$$

From (4.3), we have

Proposition 4.1: Let g be a metric and ∇ torsion free linear connection on M_{2n}, then ∇^c is a metrical to $T^2(g,\nabla)$.

References

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